# Efficiency of Set Optimization with Weighted Criteria (Nonlinear Analysis and Convex Analysis)

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## 1 Introduction

In this paper, we consider efficiency of set-valued optimization problems with weighted criteria. Let $(E, \leq)$ be an ordered topological vector space, $C$ the ordering cone in $(E, \leq)$, and assume that $C$ is a closed set. Also $C^+ = \{x^* \in E^* \mid \langle x^*, x\rangle \geq 0, \forall x \in C\}$ and we choose a *weight set* $W$, a subset of $C^+$. Let $\mathcal{A}$ be the family of all nonempty compact convex sets in $E$, and $\mathcal{B}$ a nonempty subfamily of $\mathcal{A}$. Our purpose is to consider about minimal elements of $\mathcal{B}$ with weighted criteria.

In this paper, we introduce some concepts concerned with set-limit and cone-completeness, to characterize existence of such minimal elements. Also we consider completeness of some metric space including the whole space $\mathcal{A}$.

**Definition 1.1** $\emptyset \neq A,\ B \subset E$,

$$A \leq_W^l B \overset{\text{def}}{\Longleftrightarrow} \overline{\langle z^*, A+C\rangle} \supset \langle z^*, B\rangle, \forall z^* \in W$$

$$A \leq_W^u B \overset{\text{def}}{\Longleftrightarrow} \langle z^*, A\rangle \subset \overline{\langle z^*, B-C\rangle}, \forall z^* \in W$$

**Definition 1.2** (Minimal for a Family with Weight)
$B_0$ is $(l, W)$-minimal in $\mathcal{B}$ if $B_0 \in \mathcal{B}$ and condition $B \leq_W^l B_0$ implies $B_0 \leq_W^l B$.
$B_0$ is $(u, W)$-minimal in $\mathcal{B}$ if $B_0 \in \mathcal{B}$ and condition $B \leq_W^u B_0$ implies $B_0 \leq_W^u B$.

Similarly we can define $(l, W)$-maximal and $(u, W)$-maximal. In this paper we treat only the $(l, W)$-minimal notion.

# 2 Characterization of Efficiency

**Definition 2.1** ($(l,W)$-Decreasing, $(l,W)$-Complete, $(l,W)$-Section)
A net of sets $\{A_\lambda\}$ in $\mathcal{A}$ is said to be $(l,W)$-decreasing if

$$\lambda < \lambda' \quad \Longrightarrow \quad A_{\lambda'} \leq^l_W A_\lambda$$

A subfamily $\mathcal{D} \subset \mathcal{A}$ is said to be $(l,W)$-complete if there is no $(l,W)$-decreasing net $\{D_\lambda\}$ in $\mathcal{D}$ such that

$$\mathcal{D} \subset \{A \in \mathcal{A} \mid \exists \lambda \text{ such that } A \not\leq^l_W D_\lambda\}$$

Let $A \in \mathcal{A}$ and $\mathcal{D} \subset \mathcal{A}$. Then the family

$$\mathcal{D}(A) = \{D \in \mathcal{D} \mid D \leq^l_W A\}$$

is called an $(l,W)$-section in $\mathcal{D}$

**Theorem 2.1** (Existence of $(l,W)$-minimal sets)
$\mathcal{B}$ has an $(l,W)$-minimal set if and only if $\mathcal{B}$ has a nonempty $(l,W)$-complete section

**Definition 2.2** ($W$-limit, $W$-set limit)
Let $\{a_\lambda\}_\Lambda$ be a net of $E$, $x \in E$, then

$$\lim_{\lambda}{}_{\mathrm{W}}\, a_\lambda \ni x \overset{\text{def}}{\Longleftrightarrow} \forall y^* \in W, \langle y^*, a_\lambda\rangle \to \langle y^*, x\rangle.$$

the set $\lim_{\lambda}{}_{\mathrm{W}}\, a_\lambda$ is called $W$-limit of $\{a_\lambda\}$ Also let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a net of $\mathcal{A}$, $x \in E$, then

$$\operatorname*{Lim\,inf_W}_{\lambda\in\Lambda} A_\lambda \ni x \overset{\text{def}}{\Longleftrightarrow} \exists\{a_\lambda\} \text{ such that } a_\lambda \in A_\lambda, \forall\lambda \in \Lambda \text{ and } \lim_{\lambda}{}_{\mathrm{W}}\, a_\lambda \ni x$$

$$\operatorname*{Lim\,sup_W}_{\lambda\in\Lambda} A_\lambda \ni x \overset{\text{def}}{\Longleftrightarrow} \exists\{a_{\lambda'}\} \subset \{a_\lambda\}\text{: a subnet such that } a_\lambda \in A_\lambda, \forall\lambda \in \Lambda$$
$$\text{and } \lim_{\lambda'}{}_{\mathrm{W}}\, a'_\lambda \ni x$$

these are called $W$-lower and $W$-upper limits, resp.

**Definition 2.3** ($(l,W)$ and $(u,W)$-Set limits)

$$\operatorname*{Lim\,inf^l_W}_{\lambda\in\Lambda} A_\lambda = \operatorname*{Lim\,inf_W}_{\lambda\in\Lambda}(A_\lambda + C)$$

$$\operatorname*{Lim\,inf^u_W}_{\lambda\in\Lambda} A_\lambda = \operatorname*{Lim\,inf_W}_{\lambda\in\Lambda}(A_\lambda - C)$$

$$\operatorname*{Lim\,sup^l_W}_{\lambda\in\Lambda} A_\lambda = \operatorname*{Lim\,sup_W}_{\lambda\in\Lambda}(A_\lambda + C)$$

$$\operatorname*{Lim\,sup^u_W}_{\lambda\in\Lambda} A_\lambda = \operatorname*{Lim\,sup_W}_{\lambda\in\Lambda}(A_\lambda - C)$$

**Proposition 2.1** If $\mathcal{A}_\lambda$ is $(l, W)$-decreasing then

$$A \leq_W^l A_\lambda \iff A \leq_W^l \operatorname*{Lim\,inf}^l_{W\,\lambda\in\Lambda} A_\lambda$$

**Theorem 2.2** The following are equivalent:

- $\mathcal{B}$ has an $(l, W)$-minimal set
- $\mathcal{B}$ has a nonempty $(l, W)$-complete section
- There exists $A_0 \in \mathcal{A}$ such that $\mathcal{B}(A_0) = \{B \in \mathcal{B} \mid B \leq_W^l A_0\}$ is $(l, W)$-complete
- For any $(l, W)$-decreasing net $\{B_\lambda\}$ in $\mathcal{B}$, there exists $A_0 \in \mathcal{A}$ such that $A_0 \leq_W^l \operatorname{Lim\,inf}^l_{W\,\lambda\in\Lambda} B_\lambda$

**Corollary 2.1** Let $F$ be a set-valued map from a subset $X$ of a topological space into $E$. If $X$ is compact and

$$\begin{aligned} &x_\lambda \to x_0,\ \{F(x_\lambda)\} : (l, W)\text{-decreasing} \\ &\quad \Longrightarrow F(x_0) \leq_W^l \operatorname{Lim\,inf}^l_{W\,\lambda\in\Lambda} F(x_\lambda) \end{aligned}$$

then there is an $(l, W)$-minimal set in $\{F(x) \mid x \in X\}$.

# 3 Completeness

In this section, we consider about completeness of metric space $(\mathcal{A}/\equiv_W^l, d)$. At first we define a quotient space $\mathcal{A}/\equiv_W^l$ as follows:

$$\mathcal{A}/\equiv_W^l = \{[A] \mid A \in \mathcal{A}\},$$

where $[A] = \{B \in \mathcal{A} \mid A \equiv_W^l B\}$ for each $A \in \mathcal{A}$. In this space, we define an order relation. For $[A], [B] \in \mathcal{A}/\equiv_W^l$,

$$[A] \leq_W^l [B] \overset{\text{def}}{\iff} A \leq_W^l B$$

Then $\leq_W^l$ is an order relation on $\mathcal{A}/\equiv_W^l$. Next, we define a metric on the space. For $[A]$, $[B] \in \mathcal{A}/\equiv_W^l$,

$$d([A], [B]) = \sup_{y^* \in W} |\min \langle y^*, A\rangle - \min \langle y^*, B\rangle|$$

Then $d$ is a metric on $\mathcal{A}/\equiv_W^l$.

Now we have a question. Is $d$ complete?

**Counterexample 3.1** $E = \mathbf{R}^2$, $C = \mathbf{R}^2_+$, $W = [(1,0), (0,1)]$, $A_n = \{(x_1, x_2) \in E \mid 0 \leq x_1, x_2 \leq n,\ 1 \leq x_1 x_2\}$. Then $\{[A_n]\}$ is a Cauchy sequence in $\mathcal{A}/\equiv_W^l$, but $\{[A_n]\}$ does not converges to any elements of $\mathcal{A}/\equiv_W^l$. (For example, $A_0 = \{(x_1, x_2) \in E \mid 0 \leq x_1, x_2,\ 1 \leq x_1 x_2\}$, $d(A_n, A_0) \to 0$ as $n \to \infty$)

How conditions assure the completeness? Concerning the question, we have the following two theorems.

**Theorem 3.1** $\{[A_n]\}$ is a Cauchy sequence in $\mathcal{A}/\equiv^l_W$, and there exists a compact subset $K$ of $E$ such that $A_n \subset K$ for each $n$.

**Proof.** Let $\mu_{A_n} : W \to \mathbf{R}$ defined by

$$\mu_{A_n}(y^*) := \inf_{a \in A_n} \langle y^*, a \rangle, \ y^* \in W$$

then there exists a continuous function $\mu_0 : W \to \mathbf{R}$ such that $\mu_{A_n}$ converges to $\mu_0$ uniformly on $W$. For $y^* \in W$, there exists $a_{y^*} \in K$ such that $\mu_0(y^*) = \langle y^*, a_{y^*} \rangle$. Let $A_0 := \{a_{y^*} \mid y^* \in W\}$, then $\mu_0(y^*) = \inf_{a \in A_0} \langle y^*, a \rangle = \inf_{a \in \mathrm{co} A_0} \langle y^*, a \rangle = \inf_{a \in \overline{\mathrm{co}} A_0} \langle y^*, a \rangle$. Also we have $\overline{\mathrm{co}} A_0 \in \mathcal{A}$, and then we conclude the proof. □

**Theorem 3.2** $\{[A_n]\}$ is a Cauchy sequence in $\mathcal{A}/\equiv^l_W$, and there exists a compact subset $K$ of $E$ and a sequence $\{x_n\} \subset E$ such that $x_n + A_n \subset K$ for each $n$. Assume that $C^+ - C^+ = E^*$ and $E$ is reflexive, then $\{[A_n]\}$ converges some element of $\mathcal{A}$.

**Proof.** Let $\mu_{A_n} : W \to \mathbf{R}$ defined by

$$\mu_{A_n}(y^*) := \inf_{a \in A_n} \langle y^*, a \rangle, \ y^* \in W$$

then there exists a continuous function $\mu_0 : W \to \mathbf{R}$ such that $\mu_{A_n}$ converges to $\mu_0$ uniformly on $W$. From condition $x_n + A_n \subset K$, there exists $M$ such that $|\langle y^*, x_n \rangle| \leq M$ for each $y^* \in W$ and $n$, and by assumption $C^+ - C^+ = E^*$, we have $|\langle y^*, x_n \rangle| \leq M$ for each $y^* \in E^*$ and $n$. Using uniform boundedness theorem, we have $\|x_n\| \leq M$ for each $n$. Then we can choose a subsequence $\{x_{n'}\}$ and $x_0 \in E$ such that $\{x_{n'}\}$ converges to $x_0$ weakly.

For $y^* \in W$, there exists $a_{y^*} \in K$ such that $\langle y^*, x_0 \rangle + \mu_0(y^*) = \langle y^*, a_{y^*} \rangle$. Let $A_0 := \{a_{y^*} - x_0 \mid y^* \in W\}$, then $\mu_0(y^*) = \inf_{a \in A_0} \langle y^*, a \rangle = \inf_{a \in \mathrm{co} A_0} \langle y^*, a \rangle = \inf_{a \in \overline{\mathrm{co}} A_0} \langle y^*, a \rangle$ for each $y^* \in W$. Also we have $\overline{\mathrm{co}} A_0 \in \mathcal{A}$, then we complete the proof. □

## References

[1] D. Kuroiwa, "On Weighted Criteria of Set Optimization," RIMS Kokyuroku, 2001.

[2] D. Kuroiwa, "Existence Theorems of Set Optimization with Set-Valued Maps," Journal of Informations & Optimization Sciences, to appear.

[3] D. T. Luc, "Theory of Vector Optimization," Lecture Note in Econom. and Math. Systems **319**, Springer, Berlin, 1989.